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The quantisation of non-relativistic strings with arbitrary mass and charge densities

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Abstract. We present a method for extending charged point particles into strings, within a non-relativistic framework. Our method allows a discussion of models with a non-uniform mass and charge distribution, such as strings with quark-like endpoint masses. The form factor associated with a ground-state charged string is calculated, which shows a rapid fall-off with momentum transfer. We discuss the question of the second-quantisation of such models.

1. Introduction

Relativistic strings were developed originally as models of strongly interacting quark-antiquark bound-state systems in the late 1960s. At that time, a popular model of mesons was of a quark-antiquark pair connected by a gluon flux tube, the latter playing the role of a string. In the Nambu-Goto model‡ the end quarks are ignored, and the action integral concerns only the open string surface swept out in spacetime. An important feature of this model was that it predicted a Regge-type spectrum of states, and the possibility of accounting for internal degrees of freedom in a novel way.

Subsequently, string theory underwent a profound transformation in its interpretation, and it is fair to say that most of the original motivation for 'fundamental' string theory has evaporated (this is not the case for hadronic strings based on QCD). For instance, the end quarks have gone, and there seems little or no value in a Regge-like spectrum of states when the mass gap is enormous, with conventional particle fields being regarded as manifestations of superstring ground states. Furthermore, there seems little point in requiring string theory to predict Veneziano-type scattering amplitudes when we are discussing Planck-length physics, because we have no reason to believe that such amplitudes mean anything on such scales. The proper place for these amplitudes is solely within the context of hadronic physics, where they have some phenomenological basis. Overall, the physical scale associated with fundamental strings and superstrings are quite different to those originally envisaged, being a factor 10^{-20} down on ordinary hadronic length scales. At this time it is not clear what these new strings mean in terms of phenomenology.

In this paper we take the concept of the string literally, and discuss a model for a particle extended in one dimension. Unlike most conventional string models, our

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‡ Full details of the Nambu-Goto model are given in [1].

particular type of model has been constructed to allow a continuous mass and charge distribution along the string. This presents a difficult problem if we insist on a Lorentz covariant theory, and in this paper we restrict the discussion to a non-relativistic formulation. Relativistic string models usually raise problems with time-like oscillator modes. These may be circumvented with the use of constraints, with all the well known consequences of conventional string theory. What motivates our work are some interesting questions which do not require relativity to make themselves apparent, and in this paper we restrict our attention to these questions alone. Consequently, many of the standard string topics thought essential to relativistic string theory such as conformal invariance and Virasoro algebras will not occur in our discussion.

A particularly interesting and inevitable question is the meaning of the second-quantisation of string models. What makes this a rather special problem is that from some naive ways of looking at the problem, string second-quantisation should be equivalent to particle 'third-quantisation'. Relativity is not necessary to this discussion, in the same way that it is possible to second-quantise non-relativistic Schrödinger theory, and create a non-relativistic many-particle theory. In such a theory anti-particles do not appear as a matter of course, but can to be put in by hand. We will discuss the second-quantisation of our string model towards the end of this paper.

Throughout this paper we shall deal with real bosonic string degrees of freedom $\mathbf{x}(\sigma, t)$, where $\mathbf{x} = (x^1, \dots, x^n)$ are inertial (cartesian) coordinates in n -dimensional Euclidean space, σ is the string parameter, and t is absolute time, which plays the role of a dynamical evolution parameter. We will denote partial derivatives in the usual way:

$$\dot{\mathbf{x}} \equiv \frac{\partial \mathbf{x}}{\partial t} \quad \mathbf{x}' \equiv \frac{\partial \mathbf{x}}{\partial \sigma}. \quad (1.1)$$

In conventional bosonic string theory, the action integral

$$S_{\text{B}} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \mathcal{L}(\dot{\mathbf{x}}, \mathbf{x}') \quad (1.2)$$

gives the Euler-Lagrange equations of motion

$$\partial_\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right) + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}'} \right) \quad (1.3)$$

with endpoint constraints

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}'} \right)_0 = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}'} \right)_\pi = 0. \quad (1.4)$$

These constraints are the price paid for trying to build a continuum model with a sharp cut-off at $\sigma = 0$ and $\sigma = \pi$. Actually, it was this which originally motivated our work.

Because of the constraints (1.4), it is traditional to formulate open string theory on the interval $[0, \pi]$, and in a suitable gauge to assume a particular mode expansion for the coordinates $x(\sigma)$ based on cosine functions, i.e.

$$x(\sigma) = x_0 + \sum_{n=1}^{\infty} x_n \cos(n\sigma). \quad (1.5)$$

Such an expansion has the merit of satisfying the endpoint constraints (1.4). However, this expansion generates a problem with completeness, because a representation of

the functional derivative $\delta/\delta x(\sigma)$ involves both cosine and sine terms. We expect to encounter this functional derivative in the Schrödinger coordinate representation of the momentum $p(\sigma)$ conjugate to $x(\sigma)$ when we quantise. However, on quantisation only the coefficients x_n in (1.5) become operators, the cosine terms remaining c -numbers. There is then no obvious way of arranging for the appearance of sine terms in the commutator

$$\frac{i}{\hbar} [\hat{p}(\sigma), \hat{x}(\sigma')] = \delta(\sigma - \sigma') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} [\cos(n\sigma) \cos(n\sigma') + \sin(n\sigma) \sin(n\sigma')] \quad (1.6)$$

which we would have imposed if we did not know about any endpoint constraints. The usual method of circumventing this problem is to extend the string to the unphysical interval $[-\pi, 0]$, and to restrict string states to those described by even functions of $x(\sigma)$ and $p(\sigma)$ on the interval $[-\pi, \pi]$ [1].

This problem arises really because the Nambu-Goto string model of mesons ignores the quarks at the endpoints, and with them their masses. Dynamical (i.e. physical) consistency then forces the endpoint derivatives to vanish and to remain zero, whereas mathematical completeness knows nothing about quarks, and imposes no such constraints. Our approach is to tamper with the physics of the situation, rather than move away from the quantisation condition (1.6). We note that even tiny masses at the endpoints would allow non-zero derivatives there, and we investigate this in some detail. We introduce masses over our string in a certain smooth way, which avoids confronting continuous dynamical degrees of freedom with sharp cut-offs. As a check, we find that in the limit of these masses tending to zero we recover the usual string mass spectrum and the expansion (1.5). The advantage of our approach is that it allows us to investigate the physics of strings with non-zero endpoint masses. A string with discrete non-zero endpoint masses will be referred to as a *bolas*, after the device used by South American gauchos to catch cattle†.

Actually, relativistic *bolas* strings have been considered in the past [1, 2], but with apparently little success, because of nonlinearity in the chosen models. We do our best to avoid this in our model building, and we can succeed completely in the non-relativistic case. Our approach has the great merit of utilising standard Sturm-Liouville theory, and this allows us to obtain a spectrum for the quantised non-relativistic equal-mass *bolas*, discussed in the next section. Following that, we consider the second quantisation of our model. Eventually, we are confronted with the question of what the analogue of the Feynman functional-type integral might be in the case of string degrees of freedom, if it exists. This would require us to consider a calculus using functionals as the basic variables rather than square integrable functions which are used in conventional second-quantised theories.

2. Lagrange formalism

In this section we consider a model of a non-relativistic open string propagating in a two-dimensional Newtonian spacetime. Our model allows the fixing of a mass m to each end of the string. This allows the dynamical equations of motion to support non-zero derivatives with respect to the string variable σ at the endpoints. In conventional open-string models m is set to zero, which forces the derivatives to vanish at

† A real *bolas* actually involves three masses connected to a common vertex by strings.

the ends. As discussed in the introduction, this has implications for the traditional mode expansion, because on the interval $[0, \pi]$ the coefficients of possible $\sin n\sigma$ terms must vanish for consistent dynamics.

Our dependent string variable is denoted by $x(\sigma, \tau)$, where for the open string σ is usually chosen to lie in the interval $[0, \pi]$, and τ is the dynamical evolution parameter. However, we find it very convenient to extend the range of σ to the real line, and all functions of σ are taken to have domain \mathbb{R} . Integrals over σ are denoted by

$$\langle g \rangle \equiv \int_{-\infty}^{\infty} d\sigma g(\sigma) \quad (2.1)$$

for any function g of the string variable σ , and we reserve the notation (ϕ, ψ) for the inner product of quantum state vectors. The extension of the interval $[0, \pi]$ to \mathbb{R} is no more radical than the extension to $[-\pi, \pi]$ in the standard approach to open strings [1]. Throughout this paper we choose to regard the string parameter σ as dimensionless, whereas the temporal evolution parameter t will always have the dimensions of time.

The Lagrangian we will consider is of the general form

$$L = \frac{1}{2}(\rho\dot{x}^2 - \Omega x'^2) \quad (2.2)$$

where ρ and Ω are strictly positive functions on \mathbb{R} which are assumed to fall off exponentially or faster as σ tends to $\pm\infty$. This system represents a free extended particle or string. We will consider coupling external electromagnetic fields to this object in a later section.

We may justify such an approach by noting that hadronic strings such as the Nambu-Goto model were originally regarded as abstractions from more fundamental but intractable theories such as quantum chromodynamics, and were not thought of as necessarily fundamental objects in their own right. Models of the type (2.2) should be regarded as approximations to some unspecified underlying dynamics. This is particularly the case if we have 'fundamental' Planck-scale strings in mind, where we have absolutely no idea of what lies beyond these strings. With suitable choices for the functions ρ and Ω (which are not here regarded as dynamical degrees of freedom, and are assumed independent of τ), we expect to approximate conventional string theory as closely as we wish. To what extent this is true for relativistic strings is left for a subsequent paper.

Variation of the action integral

$$S_{\text{fl}} = \int_{\tau_i}^{\tau_f} d\tau L \quad (2.3)$$

gives

$$\delta S_{\text{fl}} = [\langle \rho\dot{x}\delta x \rangle]_{\tau_i}^{\tau_f} - \int_{\tau_i}^{\tau_f} d\tau [\rho\ddot{x} - (\Omega x')']\delta x \quad (2.4)$$

from which we deduce the Euler-Lagrange equations

$$\rho\ddot{x} = (\Omega x')' \quad (2.5)$$

and a conserved total linear momentum

$$P_{\tau} = \langle \rho\dot{x} \rangle. \quad (2.6)$$

To illustrate our method, we will make a specific choice of functions ρ and Ω . Let us suppose we wished to model a true *bolas*, i.e. a string with a point mass m at each

end. If we tried to attach point masses directly to the ends of a string, we would encounter an interaction between the continuous string degrees of freedom and the discrete degrees of freedom of the end masses. We found severe problems when we tried to use Heaviside functions and Dirac delta functions directly. In particular, we encountered second-class constraints, which cause problems when we try to quantise using Poisson brackets. Our method is to avoid this by the use of sequences of non-vanishing, continuous functions which approximate these distributions, and then take appropriate limits only after the full dynamics has been worked out.

First, we define the sequence of strictly positive functions

$$f_a(\sigma) = [\pi\epsilon]^{-1/2} \exp[-(\sigma - a)^2/\epsilon] \tag{2.7}$$

which represents an approximation to the Dirac delta at $\sigma = a$. We shall normally suppress the continuous sequence label ϵ , it being understood. Then

$$\lim_{\epsilon \rightarrow 0^+} f_a(\sigma) \sim \delta(\sigma - a) \tag{2.8}$$

and

$$\langle f_a \rangle = 1. \tag{2.9}$$

Another useful sequence of functions is

$$U(\sigma) = \int_{-\infty}^{\sigma} du [f_0(u) - f_{\pi}(u)] \tag{2.10}$$

which approximates the unit function on $[0, \pi]$

$$\lim_{\epsilon \rightarrow 0^+} U(\sigma) \sim \theta(\sigma) - \theta(\sigma - \pi) \tag{2.11}$$

where $\theta(\sigma)$ is the Heaviside step function. It is clear that U is strictly positive on the real line, and has the property that

$$U' = f_0 - f_{\pi}. \tag{2.12}$$

For our strategy to work in the case of a true bolas, we expect our results to be independent of the particular choice of approximation sequence for the various distributions concerned.

Now we make a specific choice for the functions ρ and Ω ; we choose

$$\rho(\sigma) = mf_0(\sigma) + mf_{\pi}(\sigma) + \mu U(\sigma) \tag{2.13}$$

and

$$\Omega(\sigma) = \omega^2 U(\sigma). \tag{2.14}$$

In the limit $\epsilon \rightarrow 0^+$ this should be equivalent to placing discrete masses m at each end of a finite string of length π and mass density μ , with string potential constant ω^2 . Then the equations of motion (2.5) take the form

$$\begin{aligned} (mf_0 + mf_{\pi} + \mu U)\ddot{x} &= \Omega'x' + \Omega x'' \\ &= \omega^2(f_0 - f_{\pi})x' + \omega^2 Ux''. \end{aligned} \tag{2.15}$$

Now in (2.15) we may consider the 'bolas limit' $\epsilon \rightarrow 0^+$, and equate coefficients of the functions U , f_0 and f_{π} . This may be justified by integrating over various subintervals of the real line. In the limit we find the equations

$$\mu \ddot{x} = \omega^2 x'' \quad 0 < \sigma < \pi \tag{2.16}$$

$$m \ddot{x}_0 = \omega^2 x'_0 \tag{2.17}$$

$$m \ddot{x}_{\pi} = -\omega^2 x'_{\pi} \tag{2.18}$$

where $x_0 \equiv x(0, \tau)$ and $x_\pi \equiv x(\pi, \tau)$. For $\sigma < 0$ and $\sigma > \pi$ the behaviour of x is undetermined by the dynamics in the bolas limit, and may be ignored because physical quantities such as the total momentum (2.6) will only involve the interval $[0, \pi]$ in this limit.

The equations of motion (2.16)–(2.18) are essentially those found in the so-called Lamb problem [3], which is a system consisting of an infinitely long massive, vibrating string connected to a mass m at one end, which can move along a rod perpendicular to the string. This system is interesting because of its similarity to the electrodynamics of point charges coupled to the electromagnetic field. For such systems, it is necessary to discuss the boundary conditions very carefully, because the role of causality appears to be undermined. Basically, the Lamb string involves a left-moving wave coming into the mass from spatial infinity, and a right-moving wave leaving the mass. It is up to us, via the boundary conditions, to decide which of these waves is interpreted as causing the other. In our case we could regard our system as a finite non-dissipative Lamb string, with a mass at each end.

This situation is reminiscent of the Wheeler–Feynman theory of radiation [4], where the propagation of the electromagnetic field is considered to involve half of the retarded solutions and half of the advanced solution. One end of our string may be regarded as a charge and the other as the rest of the universe, with the string as the medium of communication. Yet another similar situation occurs in Feynman's path integral formulation of the polaron [5], which models a point electron coupled to the phonon vibrations in a continuous polarisable solid.

We now consider what sort of classical solutions are implied by the equations of motion (2.16)–(2.18). First, we may write the general solution to the pure wave equation (2.16) as the sum of a left-moving wave F and a right-moving wave G :

$$x(\sigma, \tau) = F\left(\tau - \frac{\sqrt{\mu}}{\omega} \sigma\right) + G\left(\tau + \frac{\sqrt{\mu}}{\omega} \sigma\right) \quad 0 < \sigma < \pi. \quad (2.19)$$

Then the conditions (2.17), (2.18) generate the coupled equations

$$\begin{aligned} D(D - \beta)G(\tau) &= -D(D + \beta)F(\tau) \\ D(D + \beta)G(\tau) &= -D(D - \beta)F(\tau - T) \end{aligned} \quad (2.20)$$

where

$$\beta \equiv \frac{\omega\sqrt{\mu}}{m} \quad T \equiv \frac{2\pi\sqrt{\mu}}{\omega} \quad D \equiv \frac{d}{d\tau}. \quad (2.21)$$

These equations may be rewritten as

$$(D + \beta)^2 F(\tau) = (D - \beta)^2 F(\tau - T) + k \quad (2.22)$$

where k is an arbitrary constant. If we write

$$F(\tau) = H(\tau) + \frac{k\tau}{\beta(4 + \beta T)} \quad (2.23)$$

we find

$$(D + \beta)^2 H(\tau) = (D - \beta)^2 H(\tau - T). \quad (2.24)$$

This is a difference-differential equation, which models the finiteness of the string, and the reflection of waves from the end masses. This sort of equation is much more

subtle than an ordinary differential equation, as it really involves derivatives of all orders. It is possible to prove that if $D^2 H$ is given, integrable, and of bounded variation on the interval $-T \leq \tau < 0$ then we may determine H in the interval $[0, T]$, and hence for any interval [6]. Once we know H then we can reconstruct F and G , and hence determine the classical motion. This gives us some reason for believing that the system can be quantised, which we show in the following sections. We note that here T is the time for a signal to propagate from one end of the string to the other and back again.

3. The Hamiltonian formalism and canonical quantisation

From the Lagrangian (2.2) we obtain the canonical momentum

$$p(\sigma) \equiv \frac{\delta \mathcal{L}}{\delta \dot{x}(\sigma)} = \rho(\sigma) \dot{x}(\sigma). \tag{3.1}$$

It is very important that the density ρ has no zeros on the finite real line. Otherwise we would encounter primary constraints, which makes the Hamiltonian analysis tedious. It is also fairly easy to set up bolas-type models which generate second-class constraints. Although these can be handled via Dirac brackets, our approach has the great merit of avoiding both sorts of constraints in the non-relativistic case. The Hamiltonian for our system is

$$H = \frac{1}{2} \langle \rho^{-1} p^2 + \Omega x'^2 \rangle \tag{3.2}$$

which, with the canonical Poisson brackets

$$\{p(\sigma), x(\sigma')\}_{\text{PB}} = -\delta(\sigma - \sigma') \tag{3.3}$$

gives Hamilton's equations of motion

$$\dot{x} = \rho^{-1} p \tag{3.4}$$

$$\dot{p} = (\Omega x')'. \tag{3.5}$$

These are equivalent to the Euler-Lagrange equations obtained in the previous section.

Canonical quantisation is implemented by replacing the Poisson bracket (3.3) with the commutator

$$[\hat{p}(\sigma), \hat{x}(\sigma')] = -i \hbar \delta(\sigma - \sigma'). \tag{3.6}$$

The total momentum operator is just

$$\hat{P}_T = \langle \hat{p} \rangle \tag{3.7}$$

which has commutation relation with the Hamiltonian

$$[\hat{P}_T, \hat{H}] = i \hbar \langle (\Omega \hat{x}')' \rangle. \tag{3.8}$$

We will require all physical states $|\Psi_1\rangle, |\Psi_2\rangle$ to satisfy the condition

$$\langle \Psi_1 | \langle (\Omega \hat{x}')' \rangle | \Psi_2 \rangle = \langle \{ \Omega(\Psi_1 | \hat{x}' | \Psi_2) \}' \rangle = 0 \tag{3.9}$$

i.e. we assume the matrix element $\langle \Psi_1 | \hat{x}'(\sigma) | \Psi_2 \rangle$ does not diverge in magnitude as σ tends to $\pm\infty$. Then total momentum conservation will hold for physical states, because the function Ω falls off rapidly as σ tends to $\pm\infty$.

A Schrödinger (functional) representation for $\hat{x}(\sigma)$ and $\hat{p}(\sigma)$ is

$$\begin{aligned}\hat{x}(\sigma) &\rightarrow x(\sigma) & x(\sigma) &\in \mathbb{R} \\ \hat{p}(\sigma) &\rightarrow -i\hbar \frac{\delta}{\delta x(\sigma)}.\end{aligned}\tag{3.10}$$

These linear operators act in the vector space of functionals of the string function x . We assume properties analogous to those found in point quantum mechanics, such as the existence of a complete basis set $\{|x\rangle\}$ of string states, with for example

$$(x|x') = \delta[x - x']\tag{3.11}$$

where the right-hand side of (3.11) is a functional delta function, and

$$\hat{x}(\sigma)|x\rangle = x(\sigma)|x\rangle.\tag{3.12}$$

If we define the wavefunctional

$$\Psi[x] \equiv (x|\Psi)\tag{3.12'}$$

then $\Psi[x]$ should satisfy the Schrödinger wavefunctional equation

$$i\hbar \frac{\partial}{\partial \tau} \Psi[x] = \frac{1}{2} \left\langle : -\rho^{-1} \hbar^2 \frac{\delta^2}{\delta x^2} + \Omega x'^2 : \right\rangle \Psi[x]\tag{3.13}$$

where the symbols $: :$ denote normal ordering, as in conventional field theory. How this may be done is explained in the next section.

4. Ladder operators

The quadratic nature of the Hamiltonian makes it likely that a relatively simple algebraic structure exists, such as is found with the harmonic oscillator. To show this, we consider the operators $\langle g\hat{p} \rangle$ and $\langle h\hat{x} \rangle$, where g and h are c -number functions of σ . These operators have the following commutation relations with the Hamiltonian:

$$\begin{aligned}[\hat{H}, \langle g\hat{p} \rangle] &= -i\hbar \langle g(\Omega \hat{x}') \rangle \\ &= -i\hbar \langle (\Omega g')' \hat{x} \rangle\end{aligned}\tag{4.1}$$

and

$$[\hat{H}, \langle h\hat{x} \rangle] = -i\hbar \langle \rho^{-1} h\hat{p} \rangle.\tag{4.2}$$

In (4.1) we have integrated by parts, which is valid because these operators will be applied within matrix elements between physical states. Now we seek a linear combination of these operators which behaves like the ladder operators in the harmonic oscillator; i.e. we try to solve for λ in the equation

$$[\hat{H}, \langle g\hat{p} + h\hat{x} \rangle] = \hbar\lambda \langle g\hat{p} + h\hat{x} \rangle.\tag{4.3}$$

This gives the necessary and sufficient conditions

$$-i\rho^{-1}h = \lambda g \quad i(\Omega g')' = -\lambda h.\tag{4.4}$$

Hence

$$(\Omega g')' = -\lambda^2 \rho g.\tag{4.5}$$

This is a second-order eigenvalue differential equation in self-adjoint form with weight function ρ , to which all the standard results of Sturm–Liouville theory can be applied. These are that the eigenvalues λ^2 are real and that the corresponding functions g_λ are elements of a complete, orthogonal set [7]. By considering the integral $\langle \rho g_\lambda^2 \rangle$ it is easy to prove that λ^2 is strictly non-negative, which is necessary for the viability of our analysis. We will assume that we have a discrete spectrum of eigenvalues throughout this paper.

If we define the operators

$$A_\lambda = \langle g_\lambda \hat{p} - i\lambda \rho g_\lambda \hat{x} \rangle \tag{4.6}$$

$$A_\lambda^\dagger = \langle g_\lambda \hat{p} + i\lambda \rho g_\lambda \hat{x} \rangle \tag{4.7}$$

for $\lambda \geq 0$, we may use the results of Sturm–Liouville theory to find the commutation relations

$$[A_\lambda, A_\nu] = \hbar(\lambda - \nu) \langle \rho g_\lambda g_\nu \rangle \tag{4.8}$$

$$[A_\lambda, A_\nu^\dagger] = \hbar(\lambda + \nu) \langle \rho g_\lambda g_\nu \rangle \tag{4.9}$$

$$[A_\lambda^\dagger, A_\nu^\dagger] = \hbar(\nu - \lambda) \langle \rho g_\lambda g_\nu \rangle. \tag{4.10}$$

First, we have to discuss the possibility that $\lambda = 0$. From (4.5) we deduce that g_0 is a constant, which we set to unity. Then

$$A_0 = A_0^\dagger = \langle \hat{p} \rangle = \hat{P}_T. \tag{4.11}$$

Also, for $\nu \neq 0$,

$$\langle \rho g_0 g_\nu \rangle = \langle \rho g_\nu \rangle = -\nu^{-2} \langle (\Omega g'_\nu)' \rangle = 0. \tag{4.12}$$

Hence \hat{P}_T commutes with A_ν and A_ν^\dagger , $\nu \neq 0$. For $\lambda, \nu \neq 0$, we have

$$\langle \rho g_\lambda g_\nu \rangle = -\nu^{-2} \langle g_\lambda (\Omega g'_\nu)' \rangle = \nu^{-2} \langle g'_\lambda \Omega g'_\nu \rangle = \lambda^{-2} \langle g'_\lambda \Omega g'_\nu \rangle. \tag{4.13}$$

For $\lambda \neq \nu$ we deduce $\langle g'_\lambda \Omega g'_\nu \rangle = 0$, which means all the commutators (4.8)–(4.10) vanish in this case. Hence, the only non-zero commutation relation is

$$[A_\lambda, A_\nu^\dagger] = 2\lambda \hbar \delta_{\lambda\nu} \langle \rho g_\lambda^2 \rangle \quad \lambda, \nu \geq 0. \tag{4.14}$$

The assumed completeness of the solutions g_λ leads us to approximate any suitable string function f by the discrete sum

$$f(\sigma) \approx \sum_\lambda \langle g_\lambda^2 \rho \rangle^{-1} \langle f g_\lambda \rho \rangle g_\lambda(\sigma) \tag{4.15}$$

and the Dirac delta by

$$\begin{aligned} \delta(\sigma - \sigma') &\approx \sum_\lambda \langle g_\lambda^2 \rho \rangle^{-1} g_\lambda(\sigma) g_\lambda(\sigma') \rho(\sigma) \\ &\approx \sum_\lambda \langle g_\lambda^2 \rho \rangle^{-1} g_\lambda(\sigma) g_\lambda(\sigma') \rho(\sigma'). \end{aligned} \tag{4.16}$$

Then the Hamiltonian may be written as

$$\begin{aligned} \hat{H} &= \frac{1}{2} \sum_\lambda \langle g_\lambda^2 \rho \rangle^{-1} A_\lambda^\dagger A_\lambda \\ &= \frac{1}{2} \langle \rho \rangle^{-1} \hat{P}_T \hat{P}_T + \sum_{\lambda > 0} \langle g_\lambda^2 \rho \rangle^{-1} A_\lambda^\dagger A_\lambda \end{aligned} \tag{4.17}$$

where we have normal ordered to eliminate the zero-point energies.

An alternative approach is to expand the string coordinates as follows:

$$\hat{x}(\sigma) = \sum_{\lambda} \hat{x}_{\lambda} g_{\lambda}(\sigma). \quad (4.18)$$

Then the Hamiltonian is given by

$$\hat{H} = \frac{\hat{P}_T^2}{2M_0} + \frac{1}{2} \sum_{\lambda > 0} \left(\frac{\hat{p}_{\lambda}^2}{M_{\lambda}} + M_{\lambda} \lambda^2 \hat{x}_{\lambda}^2 \right) \quad (4.19)$$

where

$$M_{\lambda} = \langle \rho g_{\lambda}^2 \rangle \quad (> 0 \text{ for all } \lambda) \quad (4.20)$$

$$\hat{p}(\sigma) = \sum_{\lambda} M_{\lambda}^{-1} g_{\lambda}(\sigma) \rho(\sigma) \hat{p}_{\lambda} \quad (4.21)$$

with

$$[\hat{p}_{\lambda}, \hat{x}_{\lambda'}] = -i\hbar \delta_{\lambda\lambda'}. \quad (4.22)$$

If we define the ladder operators

$$\begin{aligned} a_{\lambda} &\equiv \left(\frac{\lambda M_{\lambda}}{2\hbar} \right)^{1/2} \hat{x}_{\lambda} + \frac{i}{\sqrt{2\hbar\lambda M_{\lambda}}} \hat{p}_{\lambda} & \lambda > 0 \\ a_{\lambda}^{\dagger} &\equiv \left(\frac{\lambda M_{\lambda}}{2\hbar} \right)^{1/2} \hat{x}_{\lambda} - \frac{i}{\sqrt{2\hbar\lambda M_{\lambda}}} \hat{p}_{\lambda} & \lambda > 0 \end{aligned} \quad (4.23)$$

then the only non-zero commutation relations are

$$[a_{\lambda}, a_{\lambda'}^{\dagger}] = \delta_{\lambda\lambda'} \quad \lambda > 0. \quad (4.24)$$

We find

$$\hat{H} = \frac{\hat{P}_T^2}{2M_0} + \sum_{\lambda > 0} \lambda \hbar a_{\lambda}^{\dagger} a_{\lambda} \quad (4.25)$$

from which we may read off the energy spectrum directly. Clearly, the eigenvalue $\lambda = 0$ corresponds to free translations of the system, whereas the operators a_{λ}^{\dagger} create towers of states separated in energy by multiples of $\hbar\lambda$. Energy eigenstates of the system are characterised by two sorts of quantum numbers; one of these is the linear momentum p , which may take on any real value. There is a continuum of oscillator 'ground states' $|p\rangle$, distinguished by their linear momentum p , which are annihilated by the operators a_{λ} ;

$$a_{\lambda}|p\rangle = 0 \quad \lambda > 0. \quad (4.26)$$

Excited states of these 'ground states' are given by repeated applications of the creation operators a_{λ}^{\dagger} , and we define

$$|p, \mathbf{N}\rangle \equiv \left(\prod_{\lambda > 0} \frac{(a_{\lambda}^{\dagger})^{n_{\lambda}}}{\sqrt{n_{\lambda}!}} \right) |p\rangle \quad (4.27)$$

where $\mathbf{N} = (n_{\lambda_1}, n_{\lambda_2}, \dots)$. Clearly, $|p\rangle = |p, \mathbf{0}\rangle$. If we define the Schrödinger energy eigenfunctional

$$\Psi[x|p, \mathbf{N}] \equiv \langle x|p, \mathbf{N}\rangle \quad (4.28)$$

then

$$\Psi[x|p, \mathbf{N}](\tau) = \exp(-iE_{p, \mathbf{N}}\tau/\hbar) \Psi[x|p, \mathbf{N}] \quad (4.29)$$

is an energy eigenstate of the Schrödinger equation (3.13) with energy

$$E_{p,N} = \frac{p^2}{2M_0} + \sum_{\lambda > 0} n_\lambda \hbar \lambda. \tag{4.30}$$

The above is rather general, and we may be able to fit a variety of experimental energy levels with appropriate choice of functions ρ and Ω . For completeness, we will now find the equal-mass bolas energy spectrum. When ρ and Ω take the specific form (2.13) and (2.14) we find that (4.5) gives

$$\omega^2(f_0 - f_\pi)g' + \omega^2 U g'' = -(mf_0 + mf_\pi + \mu U)\lambda^2 g. \tag{4.31}$$

In the bolas limit, equating coefficients of f_0 , f_π , and U gives

$$\omega^2 g''(\sigma) = -\mu \lambda^2 g(\sigma) \quad 0 < \sigma < \pi \tag{4.32}$$

$$\omega^2 g'(0) = -m \lambda^2 g(0) \tag{4.33}$$

$$\omega^2 g'(\pi) = m \lambda^2 g(\pi). \tag{4.34}$$

For $\lambda \neq 0$, equation (4.32) has the solution

$$g(\sigma) = \cos\left(\frac{\lambda\sqrt{\mu}}{\omega} \sigma + k\right) \tag{4.35}$$

where we set the normalisation constant to unity. Then (4.33) and (4.34) give the conditions

$$\tan(k) = \frac{m\lambda}{\omega\sqrt{\mu}} \quad \tan\left(\frac{\lambda\sqrt{\mu}}{\omega} \pi + k\right) = -\frac{m\lambda}{\omega\sqrt{\mu}}. \tag{4.36}$$

Then λ satisfies one of two possible equations:

$$\tan\left(\frac{\lambda\sqrt{\mu} \pi}{2\omega}\right) = -\frac{m\lambda}{\omega\sqrt{\mu}} \tag{4.37}$$

$$\cot\left(\frac{\lambda\sqrt{\mu} \pi}{2\omega}\right) = \frac{m\lambda}{\omega\sqrt{\mu}}. \tag{4.38}$$

We may label solutions of (4.37) by

$$\lambda = \pm \lambda_{2n} \quad n = 0, 1, 2, \dots \tag{4.39}$$

and those of (4.38) by

$$\lambda = \pm \lambda_{2n+1} \quad n = 0, 1, 2, \dots \tag{4.40}$$

The solution $\lambda_0 = 0$ in (4.37) corresponds to the continuum ‘ground states’ discussed above.

In the limit $m \rightarrow 0$, which corresponds to the usual pure string model, we find

$$\lambda_{2n} = 2n \frac{\omega}{\sqrt{\mu}} \tag{4.41}$$

$$\lambda_{2n+1} = (2n + 1) \frac{\omega}{\sqrt{\mu}} \tag{4.42}$$

and we recover the usual cosine solutions

$$g_n(\sigma) = \cos(n\sigma) \quad n = 0, 1, 2, \dots \tag{4.43}$$

which have zero endpoint derivatives as we would expect. In the more general bolas case ($m \neq 0$) states created by various a_λ^\dagger operators need not be degenerate in energy with any others, but in the pure string all excited levels above the first are degenerate.

5. The charged extended particle

In this section we will model a charged extended particle in three spatial dimensions. First, we discuss the neutral extended particle system. This will be described by Lagrangian

$$L = \frac{1}{2} \langle \rho \dot{\mathbf{X}} \cdot \dot{\mathbf{X}} - \Omega \mathbf{X}' \cdot \mathbf{X}' \rangle \quad (5.1)$$

where $\mathbf{X}(\sigma) = (X^1, X^2, X^3)$ in standard cartesian coordinates. The equations of motion for this system are very similar to those for the one-dimensional system discussed above. An important difference is the existence of a conserved orbital angular momentum vector \mathbf{L} , given by

$$\mathbf{L} = \langle \mathbf{X} \times \mathbf{P} \rangle \quad (5.2)$$

which satisfies the standard angular momentum algebra

$$[L^i, L^j] = i \hbar \epsilon_{ijk} L^k. \quad (5.3)$$

The coupling of external electromagnetic fields is done using the minimal coupling prescription, which gives the Hamiltonian

$$H = \langle \frac{1}{2} \rho^{-1} (\mathbf{P} - q\mathbf{A}) \cdot (\mathbf{P} - q\mathbf{A}) + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' + q\phi \rangle. \quad (5.4)$$

In this equation the fields \mathbf{A} and ϕ , which are defined over a region of spacetime, interact with the string at a local level. This means that they should be regarded as functions of the string variable σ in (5.4), because they are being sampled on the string. The function q of σ is the electric charge density over the string, and integrates to give the total charge e_T of the extended particle:

$$\langle q \rangle = e_T. \quad (5.5)$$

The first-quantised string equation (3.13) now becomes

$$i \hbar \frac{\partial}{\partial t} \Psi[\mathbf{X}] = \left\langle : \frac{1}{2\rho} \left(-i \hbar \frac{\delta}{\delta \mathbf{X}} - q\mathbf{A} \right)^2 + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' + q\phi : \right\rangle \Psi[\mathbf{X}]. \quad (5.6)$$

It is clear there are potentially serious problems in this equation with the functional derivative acting on \mathbf{A} , since

$$\frac{\delta}{\delta \mathbf{X}(\sigma)} \cdot \mathbf{A}(\mathbf{X}(\sigma')) = \delta(\sigma - \sigma') \nabla \cdot \mathbf{A}. \quad (5.7)$$

This diverges in the limit $\sigma \rightarrow \sigma'$, unless we know $\nabla \cdot \mathbf{A} = 0$. Some sort of point splitting or other regularisation procedure will be required to make sense of (5.7), as in conventional quantum field theory. We will assume that this can be done, in the same way that we assumed the Hamiltonian for the free extended particle system could be normal ordered. Such problems are a recurrent feature of conventional second-quantised field theories.

Equation (5.6) is invariant under the gauge transformation

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} - \nabla\chi \\ \phi &\rightarrow \phi' = \phi + \partial_t\chi \\ \Psi[\mathbf{X}] &\rightarrow \Psi'[\mathbf{X}] = \exp(-i\langle q\chi \rangle / \hbar)\Psi[\mathbf{X}] \end{aligned} \tag{5.8}$$

where χ is a scalar function on spacetime. If we define the gauge covariant functional derivatives

$$\tilde{\mathcal{D}} \equiv \frac{\delta}{\delta \mathbf{X}} - iq\mathbf{A}/\hbar \qquad \bar{\mathcal{D}} \equiv \frac{\delta}{\delta \mathbf{X}} + iq\mathbf{A}/\hbar \tag{5.9}$$

then (5.6) becomes

$$i\hbar\partial_t\Psi[\mathbf{X}] = \left\langle -\frac{\hbar^2}{2\rho}\tilde{\mathcal{D}} \cdot \bar{\mathcal{D}} + \frac{1}{2}\Omega\mathbf{X}' \cdot \mathbf{X}' + q\phi \right\rangle\Psi[\mathbf{X}] \tag{5.10}$$

with conjugate equation

$$-i\hbar\partial_t\Psi^*[\mathbf{X}] = \Psi^*[\mathbf{X}] \left\langle -\frac{\hbar^2}{2\rho}\bar{\mathcal{D}} \cdot \tilde{\mathcal{D}} + \frac{1}{2}\Omega\mathbf{X}' \cdot \mathbf{X}' + q\phi \right\rangle. \tag{5.11}$$

A very important question is the existence of a conserved charge density and current. Whilst there is an obvious form for the charge density in the case of a single-particle theory, a suitable definition has to be contrived in the case of a many-particle system. This is because in the latter case the Born probability density $\Psi^*\Psi$ depends on as many sets of three-vector coordinates as there are particles in the system. For a string this means the analogous object depends on an infinite number of coordinates. A proper charge density $\bar{\rho}(\mathbf{x}, t)$, however, should depend on only three spatial coordinates, and should satisfy the following criteria.

(i) The total integral over space of $\bar{\rho}$ gives the total electric charge:

$$\int_{\infty} d\mathbf{x} \bar{\rho} = e_T.$$

(ii) There is a continuity equation

$$\partial_t\bar{\rho} + \nabla \cdot \mathbf{j} = 0$$

where \mathbf{j} is the electric charge current.

(iii) Both $\bar{\rho}$ and \mathbf{j} are gauge invariant.

A suitable definition of $\bar{\rho}$ and \mathbf{j} which satisfies these conditions is

$$\begin{aligned} \bar{\rho}(\mathbf{x}, t) &\equiv \int [d\mathbf{Y}] \langle q\delta^3(\mathbf{x} - \mathbf{Y}) \rangle \Psi^*[\mathbf{Y}]\Psi[\mathbf{Y}] \\ \mathbf{j}(\mathbf{x}, t) &\equiv \frac{i\hbar}{2} \int [d\mathbf{Y}] \Psi^*[\mathbf{Y}] \left\langle \tilde{\mathcal{D}} \frac{q}{\rho} \delta^3(\mathbf{x} - \mathbf{Y}) - \frac{q}{\rho} \delta^3(\mathbf{x} - \mathbf{Y}) \bar{\mathcal{D}} \right\rangle \Psi[\mathbf{Y}] \end{aligned} \tag{5.12}$$

where we functionally integrate over possible string configurations, assuming the normalisation condition

$$\int [d\mathbf{Y}] \Psi^*[\mathbf{Y}]\Psi[\mathbf{Y}] = 1. \tag{5.13}$$

These integrals may be defined either by slicing up the string, rather like a one-dimensional lattice, or by using some basis set over the space of string functions. A natural complete basis set to use is the set $\{g_\lambda\}$ introduced in earlier sections. It is not hard to prove that these densities satisfy all the required properties of a charge density and charge current.

These charge densities may be obtained directly and elegantly from the second-quantisation procedure, discussed in section 7.

6. Coulomb scattering and form factors

In this section we will consider the scattering of a charged extended particle from a fixed Coulombic potential. In general, charged particles which have some extended structure do not scatter exactly like point charges. This happens in the case of the proton and the neutron. The departure from point-like behaviour is usually described by form factors, which are functions of momentum transfer. An electrically neutral particle such as the neutron can have a non-zero form factor. We will find the relevant form factor of an extended point charge in its ground state.

Our notation in this section will be as follows. String coordinates along the string will be denoted by $\mathbf{X}(\sigma)$, with the decomposition

$$\mathbf{X}(\sigma) = \mathbf{x} + \sum_{\lambda > 0} \mathbf{y}_\lambda g_\lambda(\sigma). \quad (6.1)$$

In and out states will be taken to be of the form

$$\Psi_{p,N}[\mathbf{X}](t) = \exp(-iE_{p,N}t/\hbar + i\mathbf{p} \cdot \mathbf{x}/\hbar) \Phi_N[\mathbf{y}] \quad (6.2)$$

where $\mathbf{N} = (n_{\lambda_1}, n_{\lambda_2}, \dots)$ denotes the string oscillator excitation numbers, $\Phi_N[\mathbf{y}]$ is an oscillator wavefunctional, and the energy $E_{p,N}$ is given by

$$E_{p,N} = \frac{\mathbf{p} \cdot \mathbf{p}}{2M_0} + \sum_{\lambda > 0} (n_\lambda^1 + n_\lambda^2 + n_\lambda^3) \hbar \lambda. \quad (6.3)$$

The normalisation of the scattering states is just

$$\begin{aligned} (\Psi_{p',N'}, \Psi_{p,N}) &= \int [d\mathbf{X}] \Psi_{p',N'}^*[\mathbf{X}](t) \Psi_{p,N}[\mathbf{X}](t) \\ &= \exp[i(E_{p',N'} - E_{p,N})t/\hbar] \int d\mathbf{x} \exp[i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}/\hbar] \int [d\mathbf{y}] \Phi_{N'}^*[\mathbf{y}] \Phi_N[\mathbf{y}] \\ &= (2\pi\hbar)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{N,N'}. \end{aligned} \quad (6.4)$$

This corresponds to a particle density of one per unit volume for the incident flux in a scattering beam experiment.

We shall consider the simplest case, which involves the scattering of an extended charged particle by a pure Coulombic point-like potential. The time-dependent wavefunctional equation in this case is

$$i\hbar \partial_t \Psi[\mathbf{X}](t) = \left\langle : \frac{-\hbar^2}{2\rho} \frac{\delta^2}{\delta \mathbf{X}^2} + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' + \frac{eq}{4\pi\epsilon_0 |\mathbf{X}|} : \right\rangle \Psi[\mathbf{X}](t) \quad (6.5)$$

where the symbols $::$ denote appropriate normal ordering. In this model, the potential is due to a point charge e at the origin of coordinates. It should be possible to consider

the scattering of two strings, which is a calculation we reserve for the future. The effect of extending the source charge should be to smooth out possible divergent behaviour.

A formal solution to (6.5) is

$$\Psi[\mathbf{X}](t) = \Psi_i[\mathbf{X}](t) + \int d\tau \int [d\mathbf{Y}] G^+[\mathbf{X}, \mathbf{Y}](t, \tau) \left\langle \frac{eq}{4\pi\epsilon_0|\mathbf{Y}|} \right\rangle \Psi[\mathbf{Y}](\tau) \quad (6.6)$$

where the retarded propagator $G^+[\mathbf{X}, \mathbf{Y}](t, \tau)$ satisfies the functional equation

$$\left(i\hbar\partial_t + \frac{1}{2} \left\langle : \frac{\hbar^2}{\rho} \frac{\delta^2}{\delta\mathbf{X}^2} - \Omega\mathbf{X}' \cdot \mathbf{X}' : \right\rangle \right) G^+[\mathbf{X}, \mathbf{Y}](t, \tau) = \delta(t - \tau) \delta[\mathbf{X} - \mathbf{Y}]. \quad (6.7)$$

This has the formal solution

$$G^+[\mathbf{X}, \mathbf{Y}](t, \tau) = -\frac{i}{\hbar} \theta(t - \tau) \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \sum_N \Psi_{p,N}[\mathbf{X}](t) \Psi_{p,N}^*[\mathbf{Y}](\tau). \quad (6.8)$$

We now consider the scattering of a string from an initial state $\Psi_{p,N}$ to a final state $\Psi_{p',N'}$. The scattering matrix element S_{fi} is given by

$$S_{fi} = \lim_{t \rightarrow \infty} (\Psi_f, \Psi_i) \quad (6.9)$$

which reduces to

$$S_{fi} = (2\pi\hbar)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{N,N'} + R_{fi} \quad (6.10)$$

with the reaction matrix element R_{fi} given by

$$R_{fi} = -\frac{i}{\hbar} \int d\tau \int [d\mathbf{Y}] \Psi_{p',N'}^*[\mathbf{Y}](\tau) \left\langle \frac{eq}{4\pi\epsilon_0|\mathbf{Y}|} \right\rangle \Psi[\mathbf{Y}](\tau). \quad (6.11)$$

Making a perturbation expansion of the form

$$\Psi[\mathbf{Y}](\tau) = \Psi_{p,N}[\mathbf{Y}](\tau) + \dots \quad (6.12)$$

and neglecting all terms on the right-hand side except the first, we find

$$R_{fi} = -i2\pi\delta(E_{p,N} - E_{p',N'}) M_{fi} \quad (6.13)$$

where M_{fi} is the reduced matrix element

$$\begin{aligned} M_{fi} &= \int [d\mathbf{Y}] \Psi_{p',N'}^*[\mathbf{Y}] \left\langle \frac{eq}{4\pi\epsilon_0|\mathbf{Y}|} \right\rangle \Psi_{p,N}[\mathbf{Y}] \\ &= \int dx \int [dy] \exp[-i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x} / \hbar] \Phi_{N'}^*[y] \int d\sigma \frac{eq(\sigma)}{4\pi\epsilon_0|x+y(\sigma)|} \Phi_N[y]. \end{aligned} \quad (6.14)$$

It is easy at this stage to go the wrong way and encounter divergent integrals in M_{fi} . As a functional integral which is not a simple Gaussian integral, M_{fi} requires careful handling. We shall assume without proof that the finite answer we obtain for M_{fi} is the correct one. The key step is to do the integral over \mathbf{x} first. We find

$$M_{fi} = \frac{\hbar^2 e}{\epsilon_0 |\mathbf{p}' - \mathbf{p}|^2} \left\langle q \int [dy] \Phi_{N'}^*[y] \Phi_N[y] \exp[i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{y} / \hbar] \right\rangle. \quad (6.15)$$

The remaining functional integral is defined here in terms of the coefficients in the expansion

$$\mathbf{y}(\sigma) = \sum_{\lambda > 0} y_\lambda g_\lambda(\sigma) \quad (6.16)$$

i.e.

$$\begin{aligned} & \left\langle q \int [dy] \Phi_N^*[y] \Phi_N \exp[i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{y} / \hbar] \right\rangle \\ &= \left\langle q \prod_{\lambda > 0} \int d\mathbf{y}_\lambda \Phi_{n'_\lambda}^*(\mathbf{y}_\lambda) \Phi_{n_\lambda}(\mathbf{y}_\lambda) \exp[i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{y}_\lambda \mathbf{g}_\lambda / \hbar] \right\rangle. \end{aligned} \quad (6.17)$$

For given values of n'_λ and n_λ these integrals may be evaluated using the result

$$\begin{aligned} & \int_{-\infty}^{\infty} dx \exp(-x^2 + ipx) H_m(x) H_n(x) \\ &= \exp(-p^2/4) 2^n \sqrt{\pi} m! \left(\frac{ip}{2}\right)^{n-m} L_m^{n-m}(p^2/2) \quad n \geq m \end{aligned} \quad (6.18)$$

where H_n is a Hermite polynomial and L_m^{n-m} is an associated Laguerre polynomial. In principle this result allows a closed, albeit complicated, expression for scattering from an initial excited string state to any other excited state. If we believe that the observed elementary particles are ground-state configurations, then we will consider scattering with $\mathbf{N} = \mathbf{N}' = \mathbf{0}$ only. This leads to the result

$$\left\langle q \int [dy] \Phi_0^*[y] \Phi_0[y] \exp[i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{y} / \hbar] \right\rangle = \left\langle q \exp\left(-\frac{(\mathbf{p}' - \mathbf{p})^2}{4\hbar} \sum_{\lambda > 0} \frac{\mathbf{g}_\lambda^2}{M_{\lambda\lambda}}\right) \right\rangle. \quad (6.19)$$

Hence the full reaction matrix for ground-state elastic scattering from a Coulombic potential is

$$R_{fi} = -i2\pi\delta\left(\frac{(\mathbf{p}')^2}{2M_0} - \frac{(\mathbf{p})^2}{2M_0}\right) \frac{\hbar^2 e}{\epsilon_0(\mathbf{p}' - \mathbf{p})^2} \left\langle q \exp\left(-\frac{(\mathbf{p}' - \mathbf{p})^2}{4\hbar} \sum_{\lambda > 0} \frac{\mathbf{g}_\lambda^2}{M_{\lambda\lambda}}\right) \right\rangle. \quad (6.20)$$

From this it follows directly that the differential cross section is the standard Rutherford cross section multiplied by a rapidly decreasing function of momentum transfer squared. Essentially, the extended charged object in its ground state can be thought of as a single particle with an associated form factor

$$F(\mathbf{p}' - \mathbf{p}) = \left\langle q \exp\left(-\frac{(\mathbf{p}' - \mathbf{p})^2}{4\hbar} \sum_{\lambda > 0} \frac{\mathbf{g}_\lambda^2}{M_{\lambda\lambda}}\right) \right\rangle. \quad (6.21)$$

Taking into account the delta function in (6.20), we may write this form factor as

$$F(\mathbf{p}' - \mathbf{p}) = \int_{-\infty}^{\infty} d\sigma q(\sigma) \exp\left(-\frac{\mathbf{p}^2}{\hbar} \sin^2(\frac{1}{2}\theta) G(\sigma)\right) \quad (6.22)$$

where θ is the scattering angle and $G(\sigma)$ is the function

$$G(\sigma) = \sum_{\lambda > 0} \frac{\mathbf{g}_\lambda^2(\sigma)}{M_{\lambda\lambda}}. \quad (6.23)$$

This leads to the prediction that extended particles should scatter weakly at high energies in non-forwards directions, i.e. as \mathbf{p}^2 tends to infinity at fixed, non-zero θ , since then the exponential term provides a severe damping factor in the integral. On the other hand, for finite energy in the forwards direction ($\theta = 0$) the exponential term is unity, and the form factor is just equal to that of a point particle of total charge e_T .

7. Second-quantisation

In this section we consider the second-quantisation of the free string model presented above. The second-quantisation of relativistic strings has been considered from an operator point of view by various workers [9-11], but there are fundamental problems which we do not encounter. Most importantly, we are able to use a single time parameter for all points of our string, which cannot be done in a manifestly covariant way for relativistic strings [10] and requires light-cone coordinates. Another important difference is that our first-quantised string model has no first-class constraints, whereas the first-quantised relativistic bosonic string has an infinity of them. Both of these differences make our programme of second-quantisation much easier than its relativistic counterpart.

Our approach is to consider the wavefunctional equation

$$i\hbar\partial_t\Psi[\mathbf{X}] = \left\langle : \frac{-\hbar^2}{2\rho} \frac{\delta^2}{\delta\mathbf{X}^2} + \frac{1}{2}\Omega\mathbf{X}'^2 : \right\rangle \Psi[\mathbf{X}] \tag{7.1}$$

for the state functional in the first-quantised theory to be a classical equation of motion for a classical functional $\Psi[\mathbf{X}]$, which becomes an operator in a second-quantised theory. In our model, the dynamical degrees of freedom will be functionals rather than functions.

An appropriate classical Lagrangian is

$$L = \int [d\mathbf{X}] \mathcal{L} \\ = \int [d\mathbf{X}] \Psi^*[\mathbf{X}] \left(i\hbar\vec{\partial}_t - \left\langle : \frac{\hbar^2}{2\rho} \frac{\vec{\delta}}{\delta\mathbf{X}} \cdot \frac{\vec{\delta}}{\delta\mathbf{X}} + \frac{1}{2}\Omega\mathbf{X}' \cdot \mathbf{X}' : \right\rangle \right) \Psi[\mathbf{X}]. \tag{7.2}$$

As before, we assume that all functional integrals, functional derivatives, and functional delta functions can be expressed in terms of the complete basis set of functions $\{g_\lambda\}$ discussed earlier. Questions about the interchange of limits and integrals should be avoided by working with finite sums of terms in basis set expansions, and taking the number of terms to infinity only at the end of a calculation. This is a traditional way of dealing with functionals, which usually provides a workable answer.

The Euler-Lagrange equations of motion for Ψ and Ψ^* can be recovered in the usual way, so we move directly to a Hamiltonian formalism. We define the conjugate momenta

$$\Pi[\mathbf{X}] \equiv \frac{\partial\mathcal{L}}{\partial\partial_t\Psi[\mathbf{X}]} = i\hbar\Psi^*[\mathbf{X}] \tag{7.3}$$

$$\bar{\Pi}[\mathbf{X}] \equiv \frac{\partial\mathcal{L}}{\partial\partial_t\Psi^*[\mathbf{X}]} = 0. \tag{7.4}$$

In the language of Dirac constraint theory, these are equivalent to the primary constraints

$$\chi_1[\mathbf{X}] = \Pi[\mathbf{X}] - i\hbar\Psi^*[\mathbf{X}] \approx 0 \tag{7.5}$$

$$\chi_2[\mathbf{X}] = \bar{\Pi}[\mathbf{X}] \approx 0. \tag{7.6}$$

At this point we have to contemplate the analogue of the Poisson bracket for these degrees of freedom. In classical point mechanics, the Poisson bracket involves partial

derivatives, whereas in classical field theory we use functional derivatives. Here, we have to use a concept which for want of a better name we will refer to as an f -derivative. The f -derivative of a functional with respect to itself will be defined to be

$$\tilde{\mathcal{D}}_{\Psi[X]}\Psi[Y] = \delta[X - Y] \quad (7.7)$$

where the right-hand side is a functional delta function. We stress that we have not defined the f -derivative in terms of a limit process. If we wished to think in those terms, we would have to consider the concepts of continuity, etc in the space of all functionals, which is beyond our scope at this stage. We will use (7.7) formally, and assume that at the end of the day our manipulations are mathematically consistent. The development of such a calculus, if it exists, awaits further investigation. For the present, all the usual properties of a derivative will be assumed, such as linearity and the product rule for derivatives. For example, the f -derivative of a function of Ψ is given by

$$\tilde{\mathcal{D}}_{\Psi[X]}F\{\Psi[Y]\} = \frac{\partial F}{\partial \Psi} \delta[X - Y]. \quad (7.8)$$

With the properties

$$\tilde{\mathcal{D}}_{\Psi[X]}\Psi^*[Y] = 0 \quad (7.9)$$

$$\tilde{\mathcal{D}}_{\Pi[X]}\Pi[Y] = \delta[X - Y] \quad (7.10)$$

we define Poisson brackets by

$$\begin{aligned} \{A, B\}_{\text{PB}} \equiv & \int [d\mathbf{X}] \{A \tilde{\mathcal{D}}_{\Psi[X]} \tilde{\mathcal{D}}_{\Pi[X]} B + A \tilde{\mathcal{D}}_{\Psi^*[X]} \tilde{\mathcal{D}}_{\Pi[X]} B \\ & - B \tilde{\mathcal{D}}_{\Psi[X]} \tilde{\mathcal{D}}_{\Pi[X]} A - B \tilde{\mathcal{D}}_{\Psi^*[X]} \tilde{\mathcal{D}}_{\Pi[X]} A\}. \end{aligned} \quad (7.11)$$

Then we find

$$\{\Pi[X], \Psi[Y]\}_{\text{PB}} = -\delta[X - Y] \quad (7.12)$$

$$\{\bar{\Pi}[X], \Psi^*[Y]\}_{\text{PB}} = -\delta[X - Y] \quad (7.13)$$

as we might expect.

With these definitions, the primary constraints χ_1 and χ_2 are second class, and so we need to construct Dirac brackets and eliminate the constraints χ_i before we quantise. The only variables left in phase space are then Ψ and Ψ^* , with Dirac bracket

$$\{\Psi^*[X], \Psi[Y]\}_{\text{DB}} = \frac{i}{\hbar} \delta[X - Y]. \quad (7.14)$$

The total Hamiltonian is just

$$H_T = \int [d\mathbf{X}] \Psi^*[X] \left\langle : \frac{\hbar^2}{2\rho} \frac{\bar{\delta}}{\delta \mathbf{X}} \cdot \frac{\bar{\delta}}{\delta \mathbf{X}} + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' : \right\rangle \Psi[X] \quad (7.15)$$

which, with the above Dirac bracket, leads to the correct classical equations of motion:

$$\begin{aligned} \partial_t \Psi[X] &= \{\Psi[X], H_T\}_{\text{DB}} \\ &= -\frac{i}{\hbar} \left\langle : -\frac{\hbar^2}{2\rho} \frac{\bar{\delta}}{\delta \mathbf{X}} \cdot \frac{\bar{\delta}}{\delta \mathbf{X}} + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' : \right\rangle \Psi[X]. \end{aligned} \quad (7.16)$$

For a charged string, the appropriate Lagrangian is

$$L = \int [d\mathbf{X}] \Psi^*[\mathbf{X}] \left(i\hbar \partial_t - \left\langle : \frac{\hbar^2}{2\rho} \vec{\mathcal{D}} \cdot \vec{\mathcal{D}} + \frac{1}{2} \Omega \mathbf{X}' \cdot \mathbf{X}' + q\phi : \right\rangle \right) \Psi[\mathbf{X}]. \tag{7.17}$$

Care should be taken not to confuse the gauge-covariant functional derivative $\vec{\mathcal{D}}$ with the f -derivative $\vec{\mathcal{D}}_\psi$. The charge and current densities (5.12) may be obtained from (7.17) by the functional derivatives

$$\bar{\rho}(\mathbf{x}, t) = -\frac{\delta}{\delta\phi(\mathbf{x}, t)} \int dt L \quad \mathbf{j}(\mathbf{x}, t) = \frac{\delta}{\delta\mathbf{A}(\mathbf{x}, t)} \int dt L. \tag{7.18}$$

Quantisation of this classical system proceeds in the usual manner. Dirac brackets are replaced by commutators, so we end up with the functional operator algebra

$$[\hat{\Psi}^+[\mathbf{X}], \hat{\Psi}[\mathbf{Y}]] = -\delta[\mathbf{X} - \mathbf{Y}]. \tag{7.19}$$

In the free string theory, multi-string states may be created by repeated application of the operator $\hat{\Psi}^+[\mathbf{X}]$ onto the postulated ‘Fock space’ ground state $|\mathcal{O}\rangle$. For example, a single-string state is given by

$$|\mathbf{X}\rangle = \hat{\Psi}^+[\mathbf{X}]|\mathcal{O}\rangle \tag{7.20}$$

with normalisation

$$\langle \mathbf{X} | \mathbf{Y} \rangle = \delta[\mathbf{X} - \mathbf{Y}] \tag{7.21}$$

as we might expect (taking $\langle \mathcal{O} | \mathcal{O} \rangle = 1$). It is important not to confuse the ground state $|\mathcal{O}\rangle$ of the many-string Fock space with the ‘oscillator’ ground states $|\mathbf{p}, \mathbf{0}\rangle$ in the first-quantised single-string theory. We may refer to the former state as the ‘ f -vacuum’, as it represents empty space, with no strings at all. Similar concepts were discussed by Kaku and Kikkawa [10] in the context of relativistic bosonic strings.

We may introduce source functionals into the Lagrangian, i.e. define

$$L^J = L + \int [d\mathbf{X}] (J^*[\mathbf{X}] \Psi[\mathbf{X}] + \Psi^*[\mathbf{X}] J[\mathbf{X}]) \tag{7.22}$$

and construct Green functions in the usual way. In the presence of the source functional $J[\mathbf{X}]$, we define

$$Z^J \equiv {}_J \langle \mathcal{O} | \mathcal{O} \rangle_J. \tag{7.23}$$

Then we have, by Schwinger’s action principle extended to this situation,

$$-i\hbar \vec{\mathcal{D}}_{J^*[\mathbf{X}](t)} Z^J = {}_J \langle \mathcal{O} | \hat{\Psi}[\mathbf{X}](t) | \mathcal{O} \rangle_J \tag{7.24}$$

$$-i\hbar \vec{\mathcal{D}}_{J[\mathbf{X}](t)} Z^J = {}_J \langle \mathcal{O} | \hat{\Psi}^+[\mathbf{X}](t) | \mathcal{O} \rangle_J \tag{7.25}$$

where we use

$$\vec{\mathcal{D}}_{J[\mathbf{X}](t)} J[\mathbf{Y}](t') = \delta(t - t') \delta[\mathbf{X} - \mathbf{Y}]. \tag{7.26}$$

Other time-ordered n -string f -vacuum expectation values may be obtained in the usual way.

In the case of a free string theory, we find

$$Z'_0 \approx \exp\left(\frac{-i}{\hbar} \int dt \int [d\mathbf{X}] \int dt' \int [d\mathbf{Y}] J^*[\mathbf{X}](t) G^+[\mathbf{X}, \mathbf{Y}](t, t') J[\mathbf{Y}](t') \right) \tag{7.27}$$

where $G^+[\mathbf{X}, \mathbf{Y}](t, t')$ is the retarded propagator discussed in the previous section.

At this stage it would seem that most of the standard structure of field theory will find a ready analogue in our model. This is only because we have touched on those aspects which do not require any great mathematical development. This is particularly true for the f -derivative discussed above, which we have defined in terms of its algebraic action on simple functions of a functional. A similar approach is used in the calculus of Grassmannian variables. What remains a more interesting question is the analogue, if it exists, of a Feynman path or functional integral in this model. Presumably, this involves a vastly more difficult mathematical programme than the development of the so-called 'functional' integrals constructed in field theory. At least there we may work in terms of complete basis sets of functions, or employ Feynman's time-slicing approach. In the present situation we are dealing with functionals as the basic variables, and the meaning of an integration measure over the space of possible functionals remains unclear to us at this stage. A hypothetical integral over the space of functionals may be called an ' f -integral'. As in ordinary second-quantised field theory, it may be possible to obtain the results of such integration indirectly, by solving appropriate f -differential equations. This is exactly what we have done in the above example of the free-string model.

8. Concluding remarks

In the preceding sections we have shown that it is possible to set up a consistent quantised non-relativistic theory of strings with varying charge and mass densities. One of our predictions is that a charged ground-state string scattering from an external Coulombic field should behave as if it were a point charge with a form factor which falls very rapidly with increasing momentum transfer squared. Other aspects of the non-relativistic string interacting with external electromagnetic fields are left as exercises for the interested reader. One problem we have not yet solved is the exact energy spectrum of a bound state of a string with a Coulombic point charge.

Our attempt at second-quantisation via a functional-type approach has raised the spectre of a calculus of functionals, which we can only speculate on in this paper. In the calculus of functions, we can sometimes invoke completeness, such as occurs in Sturm-Liouville theory, to set up an operational definition of what we mean by functional differentiation, functional integration, and the functional delta function. We do not assume here that such a concept can be easily formulated in the calculus of functionals and for that reason we have made no attempt to set up an integral over functionals in this paper.

None of these issues require special relativity to make their appearance. A natural question is what the introduction of relativity does to our model. This is beyond the scope of this paper, but some points may be discussed in anticipation of the answers. First, relativity does not coexist easily with models of extended objects in general, because extension implies simultaneity, which relativity teaches us is not an intrinsic concept. For this reason, the definition and description of extended objects in field theory is a notoriously difficult programme. In relativistic string models, this problem manifests itself as the need to remove time-like oscillator modes. This is usually done by introducing constraints, which make the full structure of such systems quite a task to unravel. We do not expect the situation to be any different with models such as ours. A second problem is that relativistic strings and varying 'mass' distributions tend to involve nonlinearities, as some earlier workers found [1, 2]. Moreover, this 'mass'

distribution might not be equivalent to a true rest mass distribution at all. Our preliminary investigations have indicated that the obvious ways of introducing special relativity into our model do not achieve the anticipated goals.

If we managed to set up a relativistic theory of our charged strings, then there would be some very novel questions to answer. A very serious problem would be the electromagnetic self-interaction of what is equivalent to an infinity of point charges propagating in a coherent way as a string. Possibly, one way around this would be to argue that such a string is the effective, renormalised version of a more fundamental system, with self-interactions already taken into account. A quantum electrodynamics of charged strings and anti-strings would be a very interesting programme to investigate. A related issue is what happens at vertices where strings are created. In conventional string theory, enormous efforts are taken to maintain string sheet continuity, because it is one of the guiding tenets of string theory that string vertices provide a way of avoiding divergences. This is one of the reasons for using a Euclidean formalism instead of a Lorentzian approach in conventional string theory. Our anticipation is that charge conservation will eliminate the problem of a string splitting into three in a direct way. The strings we have in mind are not hadron-like objects which can give rise to multi-string amplitudes directly, but much more like extended particles interacting via the traditional electromagnetic field alone.

After this article was completed our attention was drawn to a series of interesting papers [8] by Nesterenko *et al* which have discussed bolas-type relativistic strings in background electromagnetic fields of various types. Their basic spectrum for a true bolas agrees with ours, but they do not discuss more arbitrary mass and charge distributions along the string. In principle, there is no reason to expect fundamental Planck-scale strings to have masses and charges only at the ends, so a more general model than the exact bolas should be considered.

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